Western Mass. ARML Summer Homework #1
SOLUTIONS / ANSWERS

The mandatory questions:

20. Find all ordered pairs (x, y) of real numbers for which $x^2+xy+x = 14$ and $y^2+xy+y = 28$.

Answer: Add the two equations to get

$$x^2 + 2xy + y^2 + x + y = 42$$

$$(x + y)^2 + (x + y) = 42$$

Let $z = x + y$. Then you have $z^2 + z - 42 = 0$, so $(z + 7)(z - 6) = 0$, which tells you that $z = -7$ or $z = 6$.

Suppose $z = 6$ (so $x + y = 6$). Then $y = 6 - x$. Plug into the first equation to get

$$x^2 + x(6 - x) + x = 14$$

$$x^2 + 6x - x^2 + x = 14$$

$$7x = 14$$

$$x = 2$$

which means $y = 6 - 2 = 4$, giving you your first answer: $\boxed{(2, 4)}$

Suppose $z = -7$ (so $x + y = -7$). Then $y = -7 - x$. Again use the first equation to get

$$x^2 + x(-7 - x) + x = 14$$

$$x^2 - 7x - x^2 + x = 14$$

$$-6x = 14$$

$$x = -\frac{7}{3}$$

which means $y = -7 - (-\frac{7}{3}) = -\frac{21}{3} + \frac{7}{3} = -\frac{14}{3}$, giving you your second and final answer: $\boxed{(-\frac{7}{3}, -\frac{14}{3})}$

2. The cube of a certain integer has a decimal representation consisting of ten digits, of which the two leftmost, as well as the rightmost, are the digit 7. Find the integer whose cube has this form.

Answer: First, narrow down the possibilities for the integer. $1000^3 = 1{,}000{,}000{,}000$ has ten digits. $2000^3 = 8{,}000{,}000{,}000$ has ten digits. Since the cube begins with 77, the number has to be between 1000 and 2000, and much closer to 2000.

Second, use the information about the last digit, and use modular arithmetic.

$$0^3 \equiv 0 \pmod{10}$$
$$1^3 \equiv 1 \pmod{10}$$
$$2^3 \equiv 8 \pmod{10}$$
$$\boxed{3^3 \equiv 7 \pmod{10}}$$
$$4^3 \equiv 4 \pmod{10}$$
$$5^3 \equiv 5 \pmod{10}$$
$$6^3 \equiv 6 \pmod{10}$$
$$7^3 \equiv 3 \pmod{10}$$
$$8^3 \equiv 2 \pmod{10}$$
$$9^3 \equiv 9 \pmod{10}$$

[Question: why is each digit in the second column represented exactly one time? It doesn't happen when the exponent in the first column is 0 or 2, but it does happen when the exponent is 1 or 3.... What is the order when the exponent is 5 or 9 or 13? Why is that?]

Since the cube ends in 7, the number ends in 3.

You don't have a calculator, so you could go through the very tedious process of cubing 1993, 1983, 1973, 1963, ..., until you find your match. Maybe it wouldn't take so long. But use algebra to save yourself some time. What's the cube of $x - 7$? Heck, make it easier than that. What's the cube of $x - 10$? $x^3 - 30x^2 + 300x - 1000$. So the cube of 1990 is $8{,}000{,}000{,}000 - 120{,}000{,}000 + 600{,}000 - 1000 = 788\ldots000$. Too big.

What's the cube of $x - 20$? $x^3 - 60x^2 + 1200x - 8000$. So the cube of 1980 is $8{,}000{,}000{,}000 - 240{,}000{,}000 + 2{,}400{,}000 - 8000 = 776\ldots000$. Just about right. You can see that 1970^3 wouldn't have two 7's at the front, given the gap between 1990^3 and 1980^3.

So your answer is $\boxed{1983.}$

4. Find the numerical value of $\cos 15°(\sin 75° + \cos 45°) + \sin 15°(\cos 75° - \sin 45°)$.

Answer: You need to know the basic angle-sum formulas for this one. Multiply out:

$$\cos 15° \sin 75° + \cos 15° \cos 45° + \sin 15° \cos 75° - \sin 15° \sin 45°$$

then rearrange to get

$$\underbrace{\cos 15° \sin 75° + \sin 15° \cos 75°}_{\sin(15°+75°)} + \underbrace{\cos 15° \cos 45° - \sin 15° \sin 45°}_{\cos(15°+45°)}$$

so you have $\sin 90° + \cos 60° = 1 + \frac{1}{2} = \boxed{\frac{3}{2}}$.

16. If x and y are real numbers, with $x > y$ and $xy = 1$, find the minimum possible value for $\frac{x^2+y^2}{x-y}$.

Answer: Look to cancel. Since you have $x - y$ on the bottom, and squares on top, complete the square:

$$\frac{x^2 - 2xy + y^2 + 2xy}{x - y}$$

$$\frac{(x - y)^2 + 2xy}{x - y}$$

$$\frac{(x - y)^2}{x - y} + \frac{2xy}{x - y}$$

You were told that $xy = 1$, and $x > y$ so $x - y \neq 0$. So you simplify to get

$$x - y + \frac{2}{x - y}$$

Let $z = x - y$, and the simplified problem is just to minimize $z + \frac{2}{z}$ $(z > 0)$.

If you don't know calculus... here's my best explanation. You want to minimize $z + \frac{2}{z}$. Since there is a minimum, suppose this minimum is met when $z = m$, so the minimum is $m + \frac{2}{m}$. That means that for any other z,

$$z + \frac{2}{z} \geq m + \frac{2}{m}.$$

Well, what does this give you?

$$z - m \geq \frac{2}{m} - \frac{2}{z}$$

$$z - m \geq \frac{2z - 2m}{mz}$$

$$z - m \geq \frac{2(z - m)}{mz}$$

If $z > m$, then $1 \geq \frac{2}{mz}$, meaning $z \geq \frac{2}{m}$.

If $z < m$, then $1 \leq \frac{2}{mz}$, meaning $z \leq \frac{2}{m}$.

But look: If z is less than m, z is less than (or equal to) $\frac{2}{m}$. And if z is greater than m, z is greater than (or equal to) $\frac{2}{m}$. The only way that can happen is if $m = \frac{2}{m}$. Thus $m^2 = 2$, so $m = \sqrt{2}$ and the minimum is $\sqrt{2} + \frac{2}{\sqrt{2}} = \boxed{2\sqrt{2}}$.

There are a lot of problems like this at `http://mathcircle.berkeley.edu/BMC4/Handouts/MaxMin.pdf`,

though without the accompanying lecture the problems aren't very accessible. Maybe I can go over this at one of the practices next year.

If you don't know calculus. . . here's the answer given in the book I got the question from. You want to minimize $z + \frac{2}{z}$. Notice that the product of these two terms is simply 2. The book asserts that you should know that if the product of two numbers is constant, to minimize their sum they must be equal. Thus $z = \frac{2}{z}$, $z^2 = 2$, $z = \sqrt{2}$.

If you know calculus. . . minimizing a one-variable equation is no problem: $(z + 2z^{-1})' = 1 - 2z^{-2} = 0$ when $z^2 = 2$, so the minimum is reached when $z = \sqrt{2}$. That means the minimum is $\sqrt{2} + \frac{2}{\sqrt{2}} = \boxed{2\sqrt{2}}$. (In general, check to make sure that's a minimum, but that's a given in this problem.)

30. A regular 11-gon is inscribed in a circle. How many triangles are there whose three vertices are all vertices of the 11-gon and whose interiors contain the center of the circle?

Answer: The first thing you should see is that there are $\binom{11}{3} = \frac{11\cdot10\cdot9}{3\cdot2\cdot1} = 165$ triangles sharing vertices with the 11-gon.

Now you have to remove the offending triangles [or count non-offending triangles, if you have a good method — anyone?].

What does it mean for a triangle to contain the center of the circle? Every side of the triangle cuts the circle into two parts. Focus on the longest side. If the last corner of the triangle is in the bigger piece of the circle, the triangle contains the center of the circle, otherwise not. (You need to draw a picture.)

So, remove the triangles that don't do that. If the longest side of the triangle has endpoints that are 2 vertices apart (9 if you went the other way), there is exactly one such triangle that doesn't contain the center. You can place that longest side in 11 different ways in the 11-gon (think rotation), so you have 11 bad triangles. If the longest side of the triangle has endpoints that are 3 vertices apart, there are 2 corresponding triangles that don't contain the center, giving you 22 bad triangles. Continue this process to find 33 bad triangles and 44 bad triangles. This gives you an answer of $165 - 11 - 22 - 33 - 44 = 165 - 110 = \boxed{55}$.

You really have to draw some pictures to understand the argument.

[The 55 is an interesting number. It implies that if you drew all 55 triangles, you could separate them into 5 piles, each with 11 similar (except for rotation) triangles. The primality of 11 ensures you don't get similar triangles after rotating less than 360°. Anyone have a good argument for this?]

The optional questions.

1. In square $ABCD$, points P, Q, R, S are chosen on sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$, respectively, so that $AP : PB = BQ : QC = CR : RD = DS : SA = 1 : 3$. Find the ratio of the area of $PQRS$ to that of $ABCD$.

   (5 : 8 or equivalent)

3. Find all ordered pairs (x, y) of real numbers such that $3^{x^2-2xy} = 1$ and $2\log_3 x = \log_3(y + 3)$.

   $((2, 1))$

5. A student guesses at random at three true-false questions. What is the probability that she gets at least two correct answers?

   $\left(\frac{1}{2}\right)$

6. In rectangle $ABCD$, $AB = 6$ and diagonal $BD = 10$. Circle O (with center O) is inscribed in triangle CBD, and circle P (with center P) is inscribed in triangle BAD. Find OP.

   $\left(2\sqrt{5}\right)$

7. For all real nonzero numbers, $f(x) = 1 - \frac{1}{x}$ and $g(x) = 1 - x$. If $h(x) = f[g(x)]$ for what value of x does $h(x) = 8$?

   $\left(x = \frac{8}{7}\right)$

8. Find the numerical value of

$$\frac{\cos 15^\circ + \sin 15^\circ}{\cos 15^\circ - \sin 15^\circ}.$$

   $\left(\sqrt{3}\right)$

9. Side $\overline{BC}$ of triangle ABC is extended through C to X so that $BC = CX$. Similarly, side $\overline{CA}$ is extended through A to Y so that $CA = AY$, and side $\overline{AB}$ is extended through B to Z so that $AB = BZ$. Find the ratio of the area of triangle XYZ to that of triangle ABC.

   (7 or 7 : 1)

10. Find the value of c for which the roots of $x^3 - 6x^2 - 24x + c = 0$ form an arithmetic progression.

    $(c = 64)$

11. Three ferryboats start at a terminal at noon and go to different destinations. Ferryboat A reaches its destination after 20 minutes, boat B after 15 minutes, and boat C after 32 minutes. Upon reaching their destinations, the boats return to the terminal,

then make another trip, and so on. The trip back to the terminal in each case is the same length, and takes the same time, as the trip out. What is the least number of hours after which the three ferries will again dock at the terminal simultaneously?

(16)

12. In right triangle ABC, leg $AC = \sin\theta$ and leg $BC = \cos\theta$. Find the length of the longer leg if the length of the median to the hypotenuse $\overline{AB}$ is $\tan\theta$.

$\left(\frac{2\sqrt{5}}{5}\right)$

13. Points M, N, and P are the respective midpoints of sides $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ of triangle ABC. A point X is chosen outside of the plane of triangle ABC. Points D, E, F are chosen such that M, N, and P are respective midpoints of $\overline{XD}$, $\overline{XE}$, and $\overline{XF}$. Find the ratio of the area of triangle DEF to that of triangle ABC.

(1)

14. For all real numbers x, the function $f(x)$ satisfies $2f(x) + f(1-x) = x^2$. Find $f(5)$.

$\left(\frac{34}{3}\right)$

15. In triangle ABC, $AB = 5$ and $AC = 8$. Point P is on $\overline{BC}$ and $BP : PC = 3 : 5$. Find the ratio of the radius of the circle through A, B, and P to the radius of the circle through A, C, and P.

$(5 : 8)$

17. If $[x]$ denotes the "greatest integer" function, find the largest prime number p such that $\left[\frac{n^2}{3}\right] = p$ for some integer n.

(5)

18. Square $ABCD$ has area 1 square unit. Point P is 5 units from its center. Set S is the set of points that can be obtained by rotating point P $90°$ counterclockwise about some point on or inside the square. Find the area of set S.

(2)

19. Circle O passes through vertex D of square $ABCD$, and is tangent to sides $\overline{AB}$ and $\overline{BC}$. If $AB = 1$, the radius of circle O can be expressed as $p + q\sqrt{2}$. Find the ordered pair of rational numbers (p, q).

$((2, -1))$

21. In a rectangular coordinate system, a tangent from the point $(24, 7)$ to the circle whose equation is $x^2 + y^2 = 400$ has point of tangency (a, b) where $b > 0$. Find a.

(12)

22. Angle ABC is a right angle and $CB = 1$. D is a point on ray BC such that $DB = 3$ and E is the point on ray BA such that $m\angle DEC$ is maximum. Find the distance BE.

$\left(\sqrt{3}\right)$

23. An ordinary pack of playing cards is shuffled, and two cards are dealt face up. Find the probability that at least one of these is a spade.

$\left(\frac{15}{34}\right)$

24. In convex quadrilateral $PQRS$, diagonals $\overline{PR}$ and $\overline{QS}$ intersect at T, with $PT : TR = 5 : 4$ and $QT : TS = 2 : 5$. Point X is chosen between T and S so that $QT = TX$, and $\overline{RX}$ is extended its own length to Y. If point Y is *outside* the quadrilateral, find the ratio of the area of triangle PSY so that of triangle QRT.

(15 : 8)

25. Point P is chosen along leg $\overline{BC}$ of right triangle ABC so that $BP = PA$. If leg $BC = 10$ and leg $AC = 4$, find BP.

$\left(\frac{29}{5}\text{ or equivalent}\right)$

26. Five identical black socks and five identical brown socks are in a drawer. Two socks are picked at random. Find the probability that the two socks picked will match.

$\left(\frac{4}{9}\right)$

27. The roots of $f(x) = 0$ are 2, 3, 7, 5, and 9.

The roots of $g(x) = 0$ are 3, 5, 7, 8, and -1.

Find all solutions of the equation $\frac{f(x)}{g(x)} = 0$.

(2, 9 (both required))

28. A set of distinct, nonzero real numbers is placed along the circumference of a circle. Each of the numbers is equal to the product of the two numbers adjacent to it. What is the least possible number of numbers in the set?

(6)

29. In equilateral triangle ABC of edge length 1, D is on $\overline{BC}$ so that $m\angle DAC = 45°$. Find the area of triangle DAC.

$\left(\frac{3-\sqrt{3}}{4}\right)$