Western Mass. ARML Summer Homework #3
SOLUTIONS / ANSWERS

The mandatory questions:

6. Find the smallest natural number N such that $\frac{N}{2}$ is a perfect square and $\frac{N}{3}$ is a perfect cube.

   Answer: First, if we are looking for the smallest natural number, the prime factors must be *only* 2 and 3. (If another prime were a factor, the number would be larger by at least a factor of p^6.)

   So the number is $2^x 3^y$.

   Because $\frac{N}{2} = \frac{2^x 3^y}{2}$ is a perfect square, we know that $x - 1$ is even and y is even.

   Because $\frac{N}{3} = \frac{2^x 3^y}{3}$ is a perfect cube, we know that x is a multiple of 3 and $y - 1$ is a multiple of 3.

   Combine the information about x: $x-1$ is even and x is a multiple of 3. The smallest x that satisfies this is $x = 3$.

   Combine the information about y: y is even and $y-1$ is a multiple of 3. The smallest y that satisfies this is $y = 4$.

   So the answer is $2^3 3^4 = 8 \cdot 81 = \boxed{648.}$

7. A circular path is 330 meters in circumference. A man makes a mark on the path, then walks around it several times, making a mark every 75 meters. He stops when the mark he is about to make coincides with his very first mark. When he is done, what is the shortest (positive) distance (measured along the circular path) between two of the marks?

   Answer: First, notice that both 330 and 75 are divisible by 15. This means that the answer cannot be any smaller than 15.

   The position of each mark can be written $75a - 330b$, were a is the number of marks made (ignoring the first mark), and b is the number of completed laps. (For example, after five marks are made, $5 \cdot 75 = 375$ meters have been traveled, or one lap and 45 meters. This can be represented as $5 \cdot 75 - 1 \cdot 330 = 45$.

   So every mark can be represented as $75a - 330b = 15(5a - 22b)$. Since a and b are integers, every mark is a multiple of 15 meters from the beginning.

   This leads to the next question: can 15 be achieved? I used trial and error: 75, 150, 225, 300, 45, 120, 195, 270, 15. Since you started at 0 and landed at 15, 15 can be achieved.

   So the answer is $\boxed{\text{15 meters.}}$

   Another way to look at it, is to change your units from meters to ARMLs, where 1 ARML equals 15 meters. Then the circular path is 22 ARMLs in circumference, and the man makes one mark every 5 ARMLs. This just makes the math easier: 0, 5, 15, 20, $25 \equiv 3$, 8, 13, 18, $23 \equiv 1$. Since you have a mark at 0 ARMLs and at 1 ARML, that must be the closest distance, and 1 ARML = $\boxed{\text{15 meters.}}$

3. How many five-digit numbers (in base ten notation, with the leftmost digit not equal to zero) are there such that each digit is strictly greater than the sum of the digits to its right (in particular, the tens digit is larger than the units digit)?

Answer: Suppose the ones digit is 0. The tens digit must be at least 1, the hundreds at least 2, the thousands at least 4, and the ten-thousands at least 8.

This gives

$$84210 \qquad \text{and} \qquad 94210$$

If the thousands were 5, the ten-thousands would have to be at least 9.

This gives

$$95210$$

If the thousands were 6, the ten-thousands would have to be at least 10, so it couldn't be a single digit.

$$x6210$$

If the hundreds were 3, the thousands would have to be at least 5, and the ten-thousands would have to be at least 10, so it couldn't be a single digit.

$$x5310$$

If the tens were 2, the hundreds would have to be at least 3, the thousands at least 6, the ten-thousands at least 12, so it couldn't be a single digit.

$$x6320$$

If the ones were 1, the tens would have to be at least 2, the hundreds at least 4, the thousands at least 8, and the ten-thousands at least 16, so it couldn't be a single digit.

$$x8421$$

So there were only $\boxed{\text{three}}$ numbers that worked: 84210, 94210, and 95210.

5. Line XY is tangent to circle O (with center O) at X and to circle P (with center P) at Y. The radii of the circles are 5 and 8, respectively, and points O and P are both on the same side of line XY. If $XY = \sqrt{7}$, find the length of OP.

Answer: This is one of those problems that is pretty easy once you see it, but is hard until then.

Draw yourself a diagram. You should know that $\angle OXY$ and $\angle XYP$ are both right angles (because a tangent to a circle meets the radius to the tangent point at a right angle).

That means $\overline{OX}$ and $\overline{PY}$ are parallel, because consecutive interior angles were supplementary. $OXYP$ is a trapezoid.

Draw a line through O, parallel to $\overline{XY}$, and label the intersection point with $\overline{PY}$ as Z. OZ divides the trapezoid into a rectangle and a right triangle. $OZ = XY = \sqrt{7}$. $ZY = OX = 5$, so $PZ = PY - ZY = 8 - 5 = 3$.

You end up with a right triangle with bases $OZ = \sqrt{7}$ and $PZ = 3$, and you want the hypotenuse OP. Well, it's $\sqrt{\sqrt{7}^2 + 3^2} = \sqrt{16} = \boxed{4.}$

11. In triangle ABC, $AB = 3$, $BC = 4$, $AC = 6$. If $\overline{BC}$ is extended through C to D and $BC = CD$, find AD.

Answer: Again, draw yourself a diagram. This is a direct application of Stewart's Theorem, which is in your formula packet.

$$3^2 4 + x^2 4 = 6^2 \cdot 8 + 4 \cdot 4 \cdot 8$$
$$9 + x^2 = 6^2 \cdot 2 + 4 \cdot 8$$
$$x^2 = 72 + 32 - 9$$
$$x^2 = 95$$
$$x = \boxed{\sqrt{95}.}$$

The optional questions.

1. The sequence $\{a_n\}$ is defined by $a_1 = a_4 = 1$, $a_2 = a_3 = a_5 = -1$, $a_6 = a_1 a_2$, $a_7 = a_2 a_3$, and in general for $k > 5$, $a_k = a_{k-5} a_{k-4}$. Find k_{1983}.

   (-1)

2. The numbers 1, 2, 3, …, 1000 are written in a row. Sam started at 1 and circled every 24th number in red. Janet started at 1 and circled every 15th number in blue. What is the smallest possible (positive) difference between a red number and a blue number?

   (3)

4. If $0 < A < \pi$, $0 < B < \pi$, and $\sin A + \sin B = \cos A + \cos B$, find the numerical value of $A + B$.

   $\left(\frac{\pi}{2}\right)$

8. In a plane, points A and B are on the same side of line L. They are each 3 cm from line L and they are 4 cm from each other. Find the radius of the circle through points A and B that is tangent to line L.

   $\left(\frac{13}{6}\right)$

9. In right triangle ABC, leg $AC = 4$ and leg $BC = 8$. A square is drawn exterior to the triangle with $\overline{AB}$ as one side. Find the distance from C to the intersection of the diagonals of the square.

   $\left(6\sqrt{2}\right)$

10. The sequence $\{a_i\}$ is defined as follows: $a_1 = 3^{1983}$, and for $i > 1$, a_i is the sum of the digits in the decimal representation of a_{i-1}. Find the numerical value of a_{10}.

    (9)

12. How many ordered pairs (x, y) of positive integers are there such that both x and y are less than 100 and the expression $\log_{10} x + \log_{10} y$ has an *integral* value?

    (16)

13. In triangle ABC, $\sin^2 A + \sin^2 B = 1$. Find the degree-measure of angle C.

    (90°)

14. The number N is represented by the base q numeral 1441. When divided by eleven, N leaves a remainder of 1. If $1 < q < 10$, find q.

    (9)

15. A certain plane geometric figure may be made to coincide with itself if it is rotated in a plane about a fixed point P through an angle of $48°$. What is the smallest positive angle through which you can be *sure* the figure may be rotated (about point P) and still coincide with itself?

    (24)

16. In triangle ABC, $AC = 6$, $BC = 8$, and $AB = 10$. A line dividing the triangle into two regions of equal area is perpendicular to $\overline{AB}$ at X. Find BX.

    $\left(4\sqrt{2} \text{ or equivalent}\right)$

17. If $m, n > 1$ and for all $x > 0$, $\log_n x = 3 \log_m x$, write an equation expressing m explicitly in terms of n.

    ($m = n^3$ (equation required))

18. In trapezoid $ABCD$, $AB = BC = CD = 6$ and $AD = 8$. Points P and R are chosen on line AD (on opposite sides of point A), so that there exists a point Q on the plane for which B is the midpoint of $\overline{PQ}$, C is the midpoint of $\overline{RQ}$, and $BQ = QC$. Find PQ.

    $\left(4\sqrt{11} \text{ or equivalent}\right)$

19. The sequence $\{a_i\}$ is defined by setting $a_1 = 7$ and, for $i > 1$, taking a_i to be the sum of the digits in the decimal representation of $(a_{i-1})^2$. Find a_{1983}.

    (16)

20. Let n be the maximum number of points that can be arranged in a plane so that of any four of them, there are three that determine an equilateral triangle. Find n.

    (5)

21. Find all values of a such that the three equations

    $$\begin{cases} ax & + & y & = & 1 \\ x & + & y & = & 2 \\ x & - & y & = & a \end{cases}$$

    are satisfied simultaneously by some ordered pair (a, b).

    (0, −1 (both required))

22. Points M and N are on side $\overline{AC}$ of triangle ABC, and points P and Q are on side $\overline{AB}$. The lines MP, NQ, and BC are parallel, and they divide the triangle into three regions of equal area. If $NQ = 4$, find BC.

    $\left(2\sqrt{6}\right)$

23. In triangle ABC, $m\angle ACB = 90$ and $m\angle ABC = 45$. Points X, Y, Z are on sides $\overline{AC}$, $\overline{CB}$, $\overline{BA}$, respectively, so that $AX : XC = CY : YB = BZ : ZA = 2 : 1$. If $\overline{CZ}$ intersects $\overline{XY}$ in P, find the degree-measure of angle CPY.

(90 or 90°)

24. The symbol $[x]$ represents the largest integer not exceeding x. Find all positive integral values of n for which the expression $\left[\frac{n^2}{3}\right]$ represents a prime number (1 is not prime).

(3, 4 (both required))

25. A sequence of digits has the property that each pair of successive digits, taken in the order written, forms a decimal numeral representing a multiple of either 17 or 23. If the first digit is 9 and the sequence is finite, what is its last (possible) digit?

(7)

26. In parallelogram $ABCD$, $AB = 4$, $AD = 9$, and $m\angle BAD = 30$. The circle through points A, B, and D intersects $\overline{BC}$ at B and X. Find XC.

$\left(4\sqrt{3}\right)$

27. A right circular cone has a base with radius 12. A plane parallel to the plane of the base cuts the cone into two equal volumes. Find, in radical form, the radius of the circle of intersection of this plane with the cone.

$\left(\frac{12}{\sqrt[3]{2}}\right)$

28. If $[x]$ denotes the greatest integer not exceeding x and $\{x\} = x - [x]$, find all ordered triples (x, y, z) of real numbers such that $x + [y] + \{z\} = 1.1$, $y + [z] + \{x\} = 2.2$, $z + [x] + \{y\} = 3.3$.

$((1, 0.2, 2.1))$

29. If $0 < x < \pi$ and $2^{\tan x} = 8^{\sin x}$, find the numerical value of $\cos x$.

$\left(\frac{1}{3}\right)$

30. Acute triangle ABC is inscribed in a circle. Altitudes $\overline{AM}$ and $\overline{CN}$ are extended to meet the circle again at P and Q, respectively. If $PQ : AC = 7 : 2$, find the numerical value of $\sin B$.

$\left(\frac{1}{4}\right)$