Western Mass. ARML Summer Homework #4
NO CALCULATORS

Send all of your answers (including your answers to any optional questions) in a single message to `rjyanco1@aol.com` by Wednesday, August 28.

These are intended to be five-minute questions (two questions in ten minutes), but take as much time as you need.

Please do these five:

4. How many ways can a set of 6 different elements be divided into 3 subsets of 2 each?

6. For *how many* natural numbers N less than or equal to 12 is $4^N + 5^N + 6^N + 7^N$ a multiple of 11?

7. Express in simplest form the real number $\left(\sqrt[3]{\sqrt{75} - \sqrt{12}}\right)^{-2}$

25. Find the largest natural number A such that x and x^3 leave the same remainder when divided by A, for any integer $x > 5$.

30. A chord of a triangle is a line segment whose endpoints lie on the sides of the triangle (but not at the vertices). For a triangle whose sides are 4, 5, 6, find the length of the shortest chord that divides the triangle into two regions of equal area.

These are optional. Do any which interest you.

1. The integer 1234321 is a perfect square. Find its positive square root.

2. Points P, Q, R, and S are chosen on sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively, of square $ABCD$ so that $AP : PB = BQ : QC = CR : RD = DS : SA = 3 : 1$. Find the ratio of the area of a square $PQRS$ to that of $ABCD$.

3. The minute hand of a clock travels $k\pi$ radians between 1 P.M. and 2:35 P.M. Find k.

5. The roots of the equation $x^2 - qx + p = 0$ are the squares of the roots of the equation $x^2 - px + q = 0$. Find the ordered pair of nonzero real numbers (p, q).

8. Line ℓ is drawn through the centroid (intersection of the medians) of triangle ABC. Points B and C are on the opposite side of line ℓ from point A. The (perpendicular) distances from A and B to ℓ are 10 and 6, respectively. Find the distance from point C to ℓ.

9. If

$$A = 1 + \cfrac{1}{1 + \frac{1}{1+\cdots}} \qquad \text{and} \qquad B = 2 + \cfrac{1}{2 + \frac{1}{2+\cdots}}$$

(where both fractions are assumed to converge), find the numerical value of

$$A + B - \left(\frac{1}{A} + \frac{1}{B}\right).$$

10. If A is an acute angle and $(\sin A)(\cos A) = \frac{60}{169}$, find the numerical value of $\sin A + \cos A$.

11. A and B both represent *nonzero* digits. If the base ten numeral AB divides (without remainder) the base ten numeral A0B (whose middle digit is zero), find all possible values for the integer AB.

12. In triangle ABC, $AB = 20$, $BC = 30$, and $\overline{BD}$ is an angle bisector (point D is on $\overline{AC}$). Point E is chosen on $\overline{BC}$ so that $\overline{DE} \parallel \overline{AB}$, and point K is chosen on $\overline{DC}$ so that $\overline{EX} \parallel \overline{BD}$. If $AD - KC = 1$, find the length of AC.

13. By using a "yardstick" that was too long, a dealer made a profit of 20% on some cloth he sold, instead of 30%. The cloth was sold by the (linear) yard. How many inches were there in his "yard" stick?

14. Points W, X, Y, and Z are chosen on sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively, of square $ABCD$ to form square $WXYZ$. If the area of square $WXYZ$ is $\frac{5}{8}$ that of $ABCD$ and $AW < WB$, find the numerical value of the ratio $AW : WB$.

15. How many fractions $\overline{a}b$ are there such that $0 < a < b$, $\frac{a}{b}$ is in lowest terms, and b divides 24 (note that b could *equal* 24)?

16. If x and y are real numbers such that $xy = 7$, find the smallest possible value of $20x^2 + 5y^2$.

17. If all numbers are written in base ten notation, how many digits are in the numeral representing 61224^2?

18. Define an interior diagonal of a three-dimensional polyhedron P to be a line segment whose endpoints are distinct vertices of P and which (except for its endpoints) lies entirely in the interior of P. Find the maximum possible number of interior diagonals if P has 10 vertices.

19. A fleeble factory was supposed to fill a certain order for fleebles in 12 days. The factory produced 25% more fleebles each day that it was supposed to, and as a result it filled the order in 10 days, producing 42 extra fleebles besides. If the same number of fleebles were produced each day, what was that number?

20. Find the value of $\arcsin \frac{5}{13} + \arccos \frac{5}{13}$, where "arc" denotes principal value.

21. If x and y are nonnegative real numbers such that $x^2 + 4y^2 = 50$, find the largest possible value of $x + 2y$.

22. In triangle ABC, D is on side $\overline{AB}$ and E is on side $\overline{AC}$. The area of triangle ADE is half that of triangle ABC, and $AD : DB = 4 : 3$. Find the ratio $AE : EC$.

23. Some children bought sticks of gum for 1¢ each, rolls for 10¢ each, and candy bars for 50¢ each. They spent \$5 and got 100 items. How many sticks of gum did they buy?

24. If $r_1, r_2, \ldots, r_5$ are the distinct roots of the equation $3x^5 + 8x^4 + 3x^3 + x^2 - 4x + 1 = 0$, find the numerical value of $(1 + r_1)(1 + r_2) \cdots (1 + r_5)$.

26. Find the area of a triangle whose sides have lengths a, b, and c if $4a^2b^2 - (a^2+b^2-c^2)^2 = 16$.

27. Find all ordered triples of integers (a, b, c) such that $0 < a < b < c$ and $a^2+b^2+c^2 = 90$.

28. If $\cos x + \cos y + \cos z = \sin x + \sin y + \sin z = 0$, find the numerical value of $\cos(x - y) + \cos(y - z) + \cos(z - x)$.

29. A cube is to be colored, with six colors, so that each face is a single different color. Two colorings are considered the same if two cubes so colored can be placed next to each other so that the colorings of *corresponding* faces are identical. Colorings that are mirror images of each other are considered distinct. How many such distinct colorings are there?